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Décembre 2007

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Abstract. In the paper, we present some modifications of the Hoede-Bakker index defined in a social network in which players may influence each other. Due to influences of the other actors, the final decision of a player may be different from his original inclination. These modifications are defined for an arbitrary probability distribution over all inclination vectors. In particular, they concern a situation in which the inclination vectors may be not equally probable. Furthermore, by assuming special probability distributions over all inclination vectors, we construct modifications of the Hoede-Bakker index that coincide with the Shapley-Shubik index and with the Holler-Packel index, respectively.

Keywords: Hoede-Bakker index, inclination vector, probability distribution, Shapley-Shubik index, Holler-Packel index

JEL Classification: C7, D7

1 Introduction

The point of departure for this paper is the concept of *decisional power* (Hoede and Bakker 1982), called also the *Hoede-Bakker index*. This concept is defined for calculating power in a social network, in which players make an acceptance-rejection decision, and they may influence each other when making such a decision. One of the main notions is the concept of an inclination vector which indicates the inclinations of all players to say 'yes' or 'no'. In Hoede and Bakker (1982) it is assumed that all inclination vectors are equally probable. In Rusinowska and De Swart (2007), some properties of the Hoede-Bakker index have been analyzed, with a focus on the postulates for power indices and the voting power paradoxes displayed by this index. In Rusinowska and De Swart (2006), the authors generalize the Hoede-Bakker index, by removing one of the originally assumed axioms. This generalization happens to coincide with the absolute Banzhaf index (Banzhaf 1965). Moreover, some modifications of the decisional power index that coincide respectively with the Rae index (Rae 1969), the Coleman indices (Coleman 1971, 1986), and the König-Bräuninger index (König and Bräuninger 1998), are proposed in Rusinowska and De Swart (2006). These modifications are defined under the assumption that all inclination vectors are equally probable.

The aim of this paper is to introduce and analyze indices, based on the Hoede-Bakker set-up, that would measure power in a social network in which inclination vectors do not have to be equally probable. First, we propose some modifications of the Hoede-Bakker index for an arbitrary probability distribution over all inclination vectors. Next, by choosing 'proper' probability distributions over all inclination vectors, we define 'a la Shapley-Shubik' and 'a la Holler-Packel' indices for a social network, that is, modifications of the decisional power index that coincide with the Shapley-Shubik index (Shapley and Shubik 1954) and modifications that coincide with the Holler-Packel index (Holler 1982, Holler and Packel 1983).

The paper is organized as follows. Section 2 concerns a probabilistic approach to power indices. In Section 3, the generalized Hoede-Bakker index is recapitulated. In Section 4, we propose several modifications of the generalized Hoede-Bakker index for an arbitrary probability distribution over the inclination vectors. Next, we choose the probability distributions under which the modifications defined become an 'a la Shapley-Shubik index' and an 'a la Holler-Packel index' for a social network. In Section 5, we present modifications of the generalized Hoede-Bakker index that coincide with the Shapley-Shubik index. Section 6 concerns a modification coinciding with the Holler-Packel index. In Section 7, an example illustrating the concepts introduced is presented. In Section 8, we conclude.

2 Probabilistic approach to power indices

There are basically two approaches to power indices: the axiomatic approach and the probabilistic one. In this research, we apply the probabilistic model for measuring 'decisiveness' in voting situations presented in Laruelle and Valenciano (2005). Below, we present the main concepts of this model.

Let $N = \{1, 2, ..., n\}$ be the set of voters who have to vote ('yes' with abstention included or 'no') on a submitted proposal. A vote configuration S is the result of voting in which all voters in S vote 'yes', and all voters in $N \setminus S$ vote 'no'. Hence, $k \in S$ means that voter k votes 'yes'. A winning configuration is a vote configuration leading to the passage of the proposal in question. Let W be the set of winning configurations. The pair (N, W) is called an N-voting rule. The following conditions are imposed: (i) $N \in W$; (ii) $\emptyset \notin W$; (iii) If $S \in W$, then $T \in W$ for any T containing S; (iv) If $S \in W$, then $N \setminus S \notin W$. [1]

A probability distribution over all possible vote configurations is assumed. It is represented by a map $p : 2^N \to [0, 1]$, associating with each vote configuration $S \subseteq N$ its probability $0 \leq p(S) \leq 1$ to occur, where $\sum_{S \subseteq N} p(S) = 1$. That is, $p(S)$ is the probability that all voters in S vote 'yes', and all voters in $N \setminus S$ vote 'no'.

Let (W, p) be an N-voting situation, where W is the voting rule to be used and p is the probability distribution over vote configurations, and let $k \in N$. Then:

$$\Phi_k(W, p) := Prob(k \ is \ decisive) = \sum_{\substack{S \,:\, k \,\in\, S \,\in\, W \\ S \setminus \{k\} \,\notin\, W}} p(S) + \sum_{\substack{S \,:\, k \,\notin\, S \,\notin\, W \\ S \cup \{k\} \,\in\, W}} p(S) =$$

$$\sum_{\substack{S \,:\, k \,\in\, S \,\in\, W \\ S \setminus \{k\} \,\notin\, W}} (p(S) + p(S \setminus \{k\})) \tag{1}$$

$$\Phi_k^+(W, p) := Prob(k \ is \ decisive \mid k \ votes \ `yes') = \frac{\sum_{\substack{S \,:\, k \,\in\, S \,\in\, W \\ S \setminus \{k\} \,\notin\, W}} p(S)}{\sum_{S : k \in S} p(S)} \tag{2}$$

[1] This condition is not necessary in this model.

$$\Phi_k^-(W,p) := Prob(k \ is \ decisive \mid k \ votes \text{ `}no') = \frac{\sum_{\substack{S\,:\,k\,\notin\,S\,\notin\,W \\ S\,\cup\,\{k\}\,\in\,W}} p(S)}{\sum_{S:k\notin S} p(S)} \tag{3}$$

3 The generalized decisional power index

The concept of *decisional power* or the *Hoede-Bakker index* (Hoede and Bakker 1982) has been generalized in Rusinowska and De Swart (2006). In this section, we recapitulate the definition of the generalized Hoede-Bakker index. We consider a social network in which $n \geq 2$ players (actors, voters) have to make an acceptance-rejection decision. Let $N = \{1, ..., n\}$ be the set of all players. Each actor has an inclination either to say 'yes' (denoted by 1) or 'no' (denoted by 0). Let i be an *inclination vector* (i.e., an n-vector consisting of ones and zeros and indicating the inclinations of the actors), and let I denote the set of all inclination vectors. Of course, $|I| = 2^n$. In the social network, players may influence each other. Due to influences of the other actors, the final decision of a player may be different from his original inclination. Each inclination vector $i \in I$ is then transformed into a *decision vector* b which is also an n-vector consisting of ones and zeros and indicating the decisions made by all players. Formally, there is an operator $B: \ I \rightarrow \ B(I)$, that is, $b = Bi$, where $B(I)$ denotes the set of all decision vectors. We also introduce the *group decision* $gd : B(I) \rightarrow \{+1, -1\}$ which is a function defined on the decision vectors b, having the value $+1$ if the group decision is 'yes', and the value -1 if the group decision is 'no'.
We introduce the following notation. Let

$$i \leq i' \iff \{k \in N \mid i_k = 1\} \subseteq \{k \in N \mid i'_k = 1\}, \tag{4}$$

and

$$i < i' \iff [i \leq i' \ \wedge \ i \neq i']. \tag{5}$$

Moreover, let $i^{\emptyset} = (0, ..., 0)$ denote an inclination vector with negative inclinations of all voters, and let $i^N = (1, ..., 1)$ denote an inclination vector with positive inclinations of all voters.

Let $gd(B)$ be the composition of B and gd. In Rusinowska and De Swart (2006), the following three conditions are imposed:

$$\forall i \in I \ \forall i' \in I \ [i \leq i' \ \Rightarrow \ gd(Bi) \leq gd(Bi')] \tag{6}$$

$$gd(Bi^N) = +1 \tag{7}$$

$$gd(Bi^{\emptyset}) = -1, \tag{8}$$

and the following definition is introduced. Given $gd(B)$, the *generalized Hoede-Bakker index* or *the generalized decisional power index of player* $k \in N$ is given by

$$GHB_k(gd(B)) = \frac{1}{2^n} \cdot \Big(\sum_{\{i:\ i_k=1\}} gd(Bi) - \sum_{\{i:\ i_k=0\}} gd(Bi)\Big). \tag{9}$$

The generalized Hoede-Bakker index is a measure of $Success - Failure$ (Rusinowska 2008), and this index measures *Decisiveness* (that is, it coincides with the absolute Banzhaf index) if all inclination vectors are equally probable (Rusinowska and De Swart 2006).

4 Modifications of the Hoede-Bakker index for an arbitrary probability distribution

In modifications of the Hoede-Bakker index proposed in Rusinowska and De Swart (2006), all inclination vectors are assumed to be equally probable, that is, the probability distribution over all inclination vectors is the following:

$$p^*(i) := \frac{1}{2^n} \ for\ all\ i \in I. \tag{10}$$

In this paper, we resign from assumption (10) and propose several modifications of the generalized Hoede-Bakker index that are defined for an arbitrary probability distribution $p : I \to [0,1]$ over all inclination vectors.

Let $k \in N$ and $i = (i_1, ..., i_n)$. By $i^k = (i_1^k, ..., i_n^k)$ we denote the inclination vector such that

$$i_j^k = \begin{cases} i_j \ if & j \neq k \\ 0 \ if \ j = k \ and \ i_j = 1 \\ 1 \ if \ j = k \ and \ i_j = 0 \end{cases}. \tag{11}$$

Similar as in Rusinowska and De Swart (2006), we introduce a bijection $f : I \to 2^N$ between inclination vectors and vote configurations, such that

$$\forall i \in I \ [f(i) = \{k \in N \mid i_k = 1\}]. \tag{12}$$

Let for each $i \in I$

$$|i| := |f(i)| = |\{k \in N \mid i_k = 1\}|. \tag{13}$$

Moreover, given $gd(B)$, we say that:

- vote configuration $f(i)$ (equivalently, inclination vector $i \in I$) is *winning* ($f(i) \in W$) iff $gd(Bi) = +1$;
- $f(i)$ (equivalently, $i \in I$) is *losing* iff $gd(Bi) = -1$;
- $f(i)$ (equivalently, $i \in I$) is *minimal winning* iff $gd(Bi) = +1$ and for each $i' < i$, $gd(Bi') = -1$.

By I^{mw} we denote the set of all minimal winning inclination vectors, that is,

$$I^{mw} := \{i \in I \mid gd(Bi) = +1 \ \wedge \ \forall i' < i \ [gd(Bi') = -1]\}. \tag{14}$$

$|I^{mw}|$ denotes the number of minimal winning inclination vectors.

We assume that the probability distribution p over all inclination vectors is the same as the probability distribution over all corresponding vote configurations, that is,

$$\forall i \in I \ [p(i) = p(f(i))], \tag{15}$$

and, consequently,

$$\sum_{i \in I} p(i) = 1, \tag{16}$$

where $0 \leq p(i) \leq 1$ denotes the probability that the inclination vector i occurs, and f is defined in (12).

We impose conditions (6), (7), and (8), and introduce the following definition.

Definition 1 *Given $gd(B)$ and probability distribution p over all inclination vectors, for each $k \in N$:*

$$\Gamma_k(gd(B), p) = \sum_{\{i:\ i_k=1\}} p(i) \cdot gd(Bi) - \sum_{\{i:\ i_k=0\}} p(i) \cdot gd(Bi). \tag{17}$$

Since $gd(Bi) \in \{-1, +1\}$ for $i \in I$, we get, for each $k \in N$, $\Gamma_k(gd(B), p) \leq 1$. Unfortunately, for an arbitrary p, $\Gamma_k(gd(B), p)$ is not a power index measuring decisiveness of player k any more, where decisiveness of k means that by changing his inclination, k also changes the group decision. In particular, $\Gamma_k(gd(B), p)$ may be even negative. Of course, by assuming some (sometimes artificial) conditions, we get $\Gamma_k(gd(B), p) \geq 0$ for $k \in N$. One may, for instance, impose a kind of 'generalized' version of (6), that is,

$$\forall i \in I\ \forall i' \in I\ [i \leq i' \ \Rightarrow\ p(i) \cdot gd(Bi) \leq p(i') \cdot gd(Bi')], \tag{18}$$

and then get $\Gamma_k(gd(B), p) \geq 0$ for each $k \in N$. Condition (18) means, in particular, that for two winning (loosing, respectively) inclination vectors i, i' such that $i \leq i'$, we have $p(i) \leq p(i')$ ($p(i) \geq p(i')$, respectively). This could suggest that the probability of voters' inclinations depends on the final results of voting.

The question arises when $\Gamma_k(gd(B), p)$ is equal to the probability that voter k is decisive. Given relation between W and $gd(B)$, that is,

$$S = f(i) \in W \ \Leftrightarrow\ gd(Bi) = +1, \tag{19}$$

we have

Proposition 1 *Given $gd(B)$, W and p, for each $k \in N$*

$$\Gamma_k(gd(B), p) = \Phi_k(W, p) \ \Leftrightarrow \sum_{\{i:\ i_k=1\}} (p(i) - p(i^k)) \cdot (gd(Bi) + gd(Bi^k)) = 0. \tag{20}$$

Proof. By virtue of (12) and (15), we have $S = f(i)$ and $p(S) = p(i)$ for each $i \in I$. Moreover, $k \in S$ iff $i_k = 1$; $k \notin S$ iff $i_k = 0$; $S \in W$ iff $gd(Bi) = +1$, $S \notin W$ iff $gd(Bi) = -1$. Hence,

($k \in S \in W$ and $S \setminus \{k\} \notin W$) iff ($i_k = 1$ and $gd(Bi) = +1$ and $gd(Bi^k) = -1$)

($k \notin S \notin W$ and $S \cup \{k\} \in W$) iff ($i_k = 0$ and $gd(Bi) = -1$ and $gd(Bi^k) = +1$)

By virtue of (1), we have then

$$\Phi_k(W, p) = \sum_{\substack{i\,:\, i_k = 1 \\ gd(Bi) = +1 \\ gd(Bi^k) = -1}} p(i) + \sum_{\substack{i\,:\, i_k = 0 \\ gd(Bi) = -1 \\ gd(Bi^k) = +1}} p(i)$$

and therefore, since (6) is imposed,

$$\Phi_k(W,p) = \sum_{\{i:\ i_k=1\}} p(i) \cdot \frac{gd(Bi) - gd(Bi^k)}{2} + \sum_{\{i:\ i_k=0\}} p(i) \cdot \frac{gd(Bi^k) - gd(Bi)}{2}. \tag{21}$$

Hence,

$$\Gamma_k(gd(B),p) = \Phi_k(W,p) \ \Leftrightarrow \ \sum_{\{i:\ i_k=1\}} p(i) \cdot gd(Bi) +$$

$$+ \sum_{\{i:\ i_k=1\}} p(i) \cdot gd(Bi^k) - \sum_{\{i:\ i_k=0\}} p(i) \cdot gd(Bi) - \sum_{\{i:\ i_k=0\}} p(i) \cdot gd(Bi^k) = 0 \ \Leftrightarrow$$

$$\sum_{\{i:\ i_k=1\}} p(i) \cdot (gd(Bi) + gd(Bi^k)) - \sum_{\{i:\ i_k=1\}} p(i^k) \cdot (gd(Bi^k) + gd(Bi)) = 0 \ \Leftrightarrow$$

$$\sum_{\{i:\ i_k=1\}} (p(i) - p(i^k)) \cdot (gd(Bi) + gd(Bi^k)) = 0.$$

□

Note that, in particular, if all inclination vectors are equally probable, that is, the probability distribution satisfies condition (10), then $\Gamma_k(gd(B),p)$ does measure decisiveness of voter k. This is, of course, consistent with the result proved in Rusinowska and De Swart (2006) that the generalized Hoede-Bakker index $GHB_k(gd(B))$ recapitulated in (9) is equal to the probability that player k is decisive.

As before, we impose conditions (6), (7), and (8), and introduce the following notations.

Definition 2 *Given $gd(B)$ and probability distribution p over all inclination vectors, for each $k \in N$:*

$$\Psi_k(gd(B),p) = \sum_{\{i:\ i_k=1\}} \frac{p(i) + p(i^k)}{2} \cdot (gd(Bi) - gd(Bi^k)) \tag{22}$$

$$\Psi_k^+(gd(B),p) = \frac{\sum_{\{i:\ i_k=1\}} p(i) \cdot (gd(Bi) - gd(Bi^k))}{2\sum_{\{i:\ i_k=1\}} p(i)} \tag{23}$$

$$\Psi_k^-(gd(B),p) = \frac{\sum_{\{i:\ i_k=0\}} p(i) \cdot (gd(Bi^k) - gd(Bi))}{2\sum_{\{i:\ i_k=0\}} p(i)}. \tag{24}$$

Of course, since $gd(Bi) \in \{-1,+1\}$ for $i \in I$, and by virtue of (6), we have for each $gd(B)$, p, and $k \in N$, $0 \le \Psi_k(gd(B),p) \le 1$, $0 \le \Psi_k^+(gd(B),p) \le 1$, and $0 \le \Psi_k^-(gd(B),p) \le 1$.

Moreover, given (19), we have

Proposition 2 *Given $gd(B)$, W and p, for each $k \in N$*

$$\Psi_k(gd(B),p) = \Phi_k(W,p) \tag{25}$$

$$\Psi_k^+(gd(B),p) = \Phi_k^+(W,p) \tag{26}$$

$$\Psi_k^-(gd(B),p) = \Phi_k^-(W,p). \tag{27}$$

Proof. By virtue of (22) and (21), we have

$$\Psi_k(gd(B),p) = \sum_{\{i:\ i_k=1\}} \frac{p(i)+p(i^k)}{2} \cdot (gd(Bi) - gd(Bi^k)) =$$

$$= \sum_{\{i:\ i_k=1\}} p(i) \cdot \frac{gd(Bi) - gd(Bi^k)}{2} + \sum_{\{i:\ i_k=1\}} p(i^k) \cdot \frac{gd(Bi) - gd(Bi^k)}{2} =$$

$$= \sum_{\{i:\ i_k=1\}} p(i) \cdot \frac{gd(Bi) - gd(Bi^k)}{2} + \sum_{\{i:\ i_k=0\}} p(i) \cdot \frac{gd(Bi^k) - gd(Bi)}{2} = \Phi_k(W,p).$$

As stated in the proof of Proposition 1, $i_k = 1$ iff $k \in S$. Moreover,
($i_k = 1$ and $gd(Bi) = +1$ and $gd(Bi^k) = -1$) iff ($k \in S \in W$ and $S \setminus \{k\} \notin W$).
Hence, by virtue of (23) and (2),

$$\Psi_k^+(gd(B),p) = \Big(\sum_{\{i:\ i_k=1\}} p(i) \cdot \frac{gd(Bi) - gd(Bi^k)}{2}\Big) \cdot \frac{1}{\sum_{\{i:\ i_k=1\}} p(i)} =$$

$$= \frac{\sum_{\substack{S\,:\,k\,\in\,S\,\in\,W \\ S\,\setminus\,\{k\}\,\notin\,W}} p(S)}{\sum_{S:k\in S} p(S)} = \Phi_k^+(W,p).$$

By analogy, $i_k = 0$ iff $k \notin S$. Moreover,
($i_k = 0$ and $gd(Bi) = -1$ and $gd(Bi^k) = +1$) iff ($k \notin S \notin W$ and $S \cup \{k\} \in W$).
Hence, by virtue of (24) and (3),

$$\Psi_k^-(gd(B),p) = \Big(\sum_{\{i:\ i_k=0\}} p(i) \cdot \frac{gd(Bi^k) - gd(Bi)}{2}\Big) \cdot \frac{1}{\sum_{\{i:\ i_k=0\}} p(i)} =$$

$$= \frac{\sum_{\substack{S\,:\,k\,\notin\,S\,\notin\,W \\ S\,\cup\,\{k\}\,\in\,W}} p(S)}{\sum_{S:k\notin S} p(S)} = \Phi_k^-(W,p).$$

□

5 Modifications leading to the Shapley-Shubik index

One of the most well-known power indices presented in the literature on voting power is the *Shapley-Shubik index* (Shapley and Shubik 1954). Using notations from Section 2, this index is defined as follows. For a given voting rule W, the Shapley-Shubik index, for each $k \in N$, is given by

$$Sh_k(W) = \sum_{\substack{S\,:\,k\,\in\,S\,\in\,W \\ S\,\setminus\,\{k\}\,\notin\,W}} \frac{(n-|S|)! \cdot (|S|-1)!}{n!}, \tag{28}$$

where $|S|$ denotes the number of voters with vote ‘yes’.

We would like to extend work done in Rusinowska and De Swart (2006), in which modifications of the generalized Hoede-Bakker index that coincide with the Rae index, the Coleman indices, and the König-Bräuninger index are presented. The question appears whether by choosing 'proper' probability distributions over all inclination vectors, our modifications of the generalized Hoede-Bakker index may coincide with the Shapley-Shubik index.

Let for $x \in N$

$$Even(x, x+1) = \begin{cases} x & if\ x\ is\ even \\ x+1 & if\ x\ is\ odd \end{cases} \tag{29}$$

and

$$Odd(x, x+1) = \begin{cases} x & if\ x\ is\ odd \\ x+1 & if\ x\ is\ even \end{cases}. \tag{30}$$

Given (19), we have

Proposition 3 *$\Psi_k(gd(B), p) = Sh_k(W)$ for all $k \in N$, $gd(B)$ and W if and only if*

$$\frac{1}{n+1} \cdot (1 - \frac{1}{\binom{n}{Even([\frac{n}{2}],[\frac{n}{2}]+1)}}) \leq p(i^{\emptyset}) \leq \frac{1}{n+1} \cdot (\frac{1}{\binom{n}{Odd([\frac{n}{2}],[\frac{n}{2}]+1)}} + 1), \tag{31}$$

and

$$p(i) = \begin{cases} \frac{1}{n+1} \cdot (\frac{1}{\binom{n}{|i|}} - 1) + p(i^{\emptyset})\ if\ |i|\ is\ even \\ \frac{1}{n+1} \cdot (\frac{1}{\binom{n}{|i|}} + 1) - p(i^{\emptyset})\ if\ |i|\ is\ odd \end{cases}. \tag{32}$$

In particular, $\Psi_k(gd(B), p) = Sh_k(W)$ for all $k \in N$, $gd(B)$ and W if for each $i \in I$

$$p(i) = \frac{1}{n+1} \cdot \frac{1}{\binom{n}{|i|}}. \tag{33}$$

Proof. By virtue of Laruelle and Valenciano (2005), $Sh_k(W) = \Phi_k(W, p)$ for all k and W if

$$p(S) = \frac{1}{n+1} \cdot \frac{1}{\binom{n}{|S|}}\ for\ S \subseteq N. \tag{34}$$

Hence, applying (25) and (15), and replacing in (34) vote configuration S by inclination vector i, and $|S|$ by $|i|$ (see (13)), we have for each $k \in N$, $\Psi_k(gd(B), p) = \Phi_k(W, p) = Sh_k(W)$, where p is defined in (33).

Moreover, by virtue of Laruelle and Valenciano (2005), if p and p' are two probability distributions, then $\Phi_k(W, p) = \Phi_k(W, p')$ for every $k \in N$ and every voting rule W iff for all $S \neq \emptyset$

$$p'(S) = p(S) + (-1)^{|S|+1} \cdot (p(\emptyset) - p'(\emptyset)). \tag{35}$$

Hence, $\Psi_k(gd(B), p) = \Psi_k(gd(B), p')$ for every $k \in N$ and $gd(B)$ iff for each $i \neq i^{\emptyset}$

$$p'(i) = p(i) + (-1)^{|i|+1} \cdot (p(i^{\emptyset}) - p'(i^{\emptyset})). \tag{36}$$

By combining (33) and (36), we get (32). Taking into account that $0 \leq p(i) \leq 1$ for each $i \in I$, and applying this to (32), we get, for each $i \in I$ such that $|i|$ is even,

$$0 \leq \frac{1}{n+1} \cdot (1 - \frac{1}{\binom{n}{|i|}}) \leq p(i^{\emptyset}) \leq 1 - \frac{1}{n+1} \cdot (\frac{1}{\binom{n}{|i|}} - 1),$$

and for each $i \in I$ such that $|i|$ is odd,

$$\frac{1}{n+1}\cdot(\frac{1}{\binom{n}{|i|}}+1)-1 \leq p(i^{\emptyset}) \leq \frac{1}{n+1}\cdot(\frac{1}{\binom{n}{|i|}}+1) \leq 1.$$

Note that for each $i \in I$,

$$1-\frac{1}{n+1}\cdot(\frac{1}{\binom{n}{|i|}}-1) \geq 1,$$

and

$$\frac{1}{n+1}\cdot(\frac{1}{\binom{n}{|i|}}+1)-1 \leq 0.$$

Hence,

$$\max_{|i|-even}\frac{1}{n+1}\cdot(1-\frac{1}{\binom{n}{|i|}}) \leq p(i^{\emptyset}) \leq \min_{|i|-odd}\frac{1}{n+1}\cdot(\frac{1}{\binom{n}{|i|}}+1),$$

and since

$$\max_{|i|-even}\frac{1}{n+1}\cdot(1-\frac{1}{\binom{n}{|i|}}) = \frac{1}{n+1}\cdot(1-\frac{1}{\binom{n}{Even([\frac{n}{2}],[\frac{n}{2}]+1)}})$$

and

$$\min_{|i|-odd}\frac{1}{n+1}\cdot(\frac{1}{\binom{n}{|i|}}+1) = \frac{1}{n+1}\cdot(\frac{1}{\binom{n}{Odd([\frac{n}{2}],[\frac{n}{2}]+1)}}+1),$$

we get (31). Let us write (32) equivalently as

$$p(i) = \frac{1}{n+1}\cdot\frac{1}{\binom{n}{|i|}}+(-1)^{|i|+1}\cdot(\frac{1}{n+1}-p(i^{\emptyset}))$$

for each $i \in I$. Note that, of course,

$$\sum_{i\in I}p(i) = \sum_{i\in I}[\frac{1}{n+1}\cdot\frac{1}{\binom{n}{|i|}}+(-1)^{|i|+1}\cdot(\frac{1}{n+1}-p(i^{\emptyset}))] =$$

$$\frac{1}{n+1}\cdot\sum_{i\in I}\frac{1}{\binom{n}{|i|}}+(\frac{1}{n+1}-p(i^{\emptyset}))\cdot\sum_{i\in I}(-1)^{|i|+1} =$$

$$= \frac{1}{n+1}\cdot\sum_{|i|=0}^{|i|=n}\frac{1}{\binom{n}{|i|}}\cdot\binom{n}{|i|}+(\frac{1}{n+1}-p(i^{\emptyset}))\cdot\sum_{|i|=0}^{|i|=n}(-1)^{|i|+1}\cdot\binom{n}{|i|} = 1.$$

□

Proposition 4 *$\Psi_k^+(gd(B),p) = Sh_k(W)$ for all $k \in N$, $gd(B)$ and W iff $0 \leq p(i^{\emptyset}) < 1$ and for any $i \neq i^{\emptyset}$*

$$p(i) = (1-p(i^{\emptyset}))\cdot\frac{\frac{1}{|i|}}{\sum_{t=1}^{n}\frac{1}{t}}\cdot\frac{1}{\binom{n}{|i|}}. \tag{37}$$

In particular, $\Psi_k^+(gd(B),p) = Sh_k(W)$ for $k \in N$, $gd(B)$ and W if $p(i^{\emptyset}) = 0$ and for any $i \neq i^{\emptyset}$

$$p(i) = \frac{\frac{1}{|i|}}{\sum_{t=1}^{n}\frac{1}{t}}\cdot\frac{1}{\binom{n}{|i|}}. \tag{38}$$

Proof. By virtue of Laruelle and Valenciano (2005), $Sh_k(W) = \Phi_k^+(W,p)$ for all k and W if $p(\emptyset) = 0$ and

$$p(S) = \frac{\frac{1}{|S|}}{\sum_{t=1}^{n}\frac{1}{t}} \cdot \frac{1}{\binom{n}{|S|}} \ for\ S \neq \emptyset. \tag{39}$$

Hence, applying (26) and (15), and replacing S by i in (39), we get, for each $k \in N$, $\Psi_k^+(gd(B),p) = \Phi_k^+(W,p) = Sh_k(W)$, where p is defined in (38).

Moreover, if p and p' are two probability distributions, then (Laruelle and Valenciano 2005) $\Phi_k^+(W,p) = \Phi_k^+(W,p')$ for every $k \in N$ and every voting rule W iff for all $S \neq \emptyset$

$$\frac{p(S)}{1-p(\emptyset)} = \frac{p'(S)}{1-p'(\emptyset)}, \tag{40}$$

where $p(\emptyset), p'(\emptyset) < 1$. Hence, we get (37). Note that, of course,

$$\sum_{i\in I} p(i) = p(i^{\emptyset}) + \sum_{i\in I\setminus\{i^{\emptyset}\}} p(i) = p(i^{\emptyset}) + (1-p(i^{\emptyset}))\cdot \sum_{i\in I\setminus\{i^{\emptyset}\}} \frac{\frac{1}{|i|}}{\sum_{t=1}^{n}\frac{1}{t}}\cdot\frac{1}{\binom{n}{|i|}} =$$

$$= p(i^{\emptyset}) + \frac{1-p(i^{\emptyset})}{\sum_{t=1}^{n}\frac{1}{t}}\cdot\sum_{|i|=1}^{|i|=n}\frac{1}{|i|}\cdot\frac{1}{\binom{n}{|i|}}\cdot\binom{n}{|i|} = 1.$$

□

Proposition 5 *$\Psi_k^-(gd(B),p) = Sh_k(W)$ for all $k \in N$, $gd(B)$ and W iff $0 \leq p(i^N) < 1$ and for any $i \neq i^N$*

$$p(i) = (1-p(i^N))\cdot\frac{\frac{1}{n-|i|}}{\sum_{t=1}^{n}\frac{1}{t}}\cdot\frac{1}{\binom{n}{|i|}}. \tag{41}$$

In particular, $\Psi_k^-(gd(B),p) = Sh_k(W)$ for $k \in N$, $gd(B)$ and W if $p(i^N) = 0$ and for any $i \neq i^N$

$$p(i) = \frac{\frac{1}{n-|i|}}{\sum_{t=1}^{n}\frac{1}{t}}\cdot\frac{1}{\binom{n}{|i|}}. \tag{42}$$

Proof. By virtue of Laruelle and Valenciano (2005), $Sh_k(W) = \Phi_k^-(W,p)$ for all k and W if $p(N) = 0$ and

$$p(S) = \frac{\frac{1}{n-|S|}}{\sum_{t=1}^{n}\frac{1}{t}}\cdot\frac{1}{\binom{n}{|S|}} \ for\ S \neq N. \tag{43}$$

Hence, applying (27) and (15), and replacing S by i in (43), we get, for each $k \in N$, $\Psi_k^-(gd(B),p) = \Phi_k^-(W,p) = Sh_k(W)$, where p is defined in (42).

Moreover, if p and p' are two probability distributions, then (Laruelle and Valenciano 2005) $\Phi_k^-(W,p) = \Phi_k^-(W,p')$ for every $k \in N$ and every voting rule W iff for all $S \neq \emptyset$

$$\frac{p(S\setminus\{k\})}{1-p(N)} = \frac{p'(S\setminus\{k\})}{1-p'(N)},$$

where $p(N), p'(N) < 1$. Hence, we get (41). Moreover, note that,

$$\sum_{i \in I} p(i) = p(i^N) + \sum_{i \in I \setminus \{i^N\}} p(i) = p(i^N) + (1 - p(i^N)) \cdot \sum_{i \in I \setminus \{i^N\}} \frac{\frac{1}{n-|i|}}{\sum_{t=1}^{n} \frac{1}{t}} \cdot \frac{1}{\binom{n}{|i|}} =$$

$$= p(i^N) + \frac{1 - p(i^N)}{\sum_{t=1}^{n} \frac{1}{t}} \cdot \sum_{|i|=0}^{|i|=n-1} \frac{1}{n - |i|} \cdot \frac{1}{\binom{n}{|i|}} \cdot \binom{n}{|i|} = 1.$$

□

6 Modification leading to the Holler-Packel index

Another power index analyzed in the literature is the Holler-Packel index (Holler 1982, Holler and Packel 1983), an index referring to *minimal winning configurations*. A winning configuration is *minimal* if it does not contain properly any other winning configuration. The Holler-Packel index is based on the following assumptions: only minimal winning configurations will be formed, all minimal winning configurations are equally probable, all voters in a minimal winning coalition get the undivided coalition value.

Let $M(W)$ be the set of all minimal winning configurations. Let $m(W)$ and $m_k(W)$ denote the number of minimal winning configurations and the number of minimal winning configurations containing voter $k \in N$, respectively. For a given voting rule W, the *non-normalized Holler-Packel index*, for each $k \in N$, is defined by

$$HP_k(W) = \frac{m_k(W)}{m(W)}, \tag{44}$$

and the *Holler-Packel index*, for each $k \in N$, is given by

$$\widetilde{HP}_k(W) = \frac{HP_k(W)}{\sum_{j \in N} HP_j(W)} = \frac{m_k(W)}{\sum_{j \in N} m_j(W)}. \tag{45}$$

One of our modifications of the generalized decisional power index coincides with the non-normalized Holler-Packel index. We have

Proposition 6 *$\Psi_k(gd(B), p^{mw}) = HP_k(W)$ for all $k \in N$, $gd(B)$ and W, where for each $i \in I$*

$$p^{mw}(i) = \begin{cases} \frac{1}{|I^{mw}|} & if\ i \in I^{mw} \\ 0 & if\ i \notin I^{mw} \end{cases}. \tag{46}$$

Proof. We have: $S = f(i) \in M(W)$ iff $i \in I^{mw}$. Applying a result from Laruelle and Valenciano (2005) that $HP_k(W) = \Phi_k(W, p_W)$ for each $k \in N$ and W, where

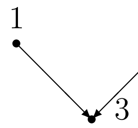
$$p_W(S) = \begin{cases} \frac{1}{m(W)} & if\ S \in M(W) \\ 0 & if\ S \notin M(W) \end{cases}, \tag{47}$$

and by virtue of (25), we have, for each $k \in N$, $\Psi_k(gd(B), p^{mw}) = \Phi_k(W, p_W) = HP_k(W)$, where $p^{mw}(i)$ is defined in (46). □

7 Example

Let us consider a very simple example of a three-voter social network presented in Figure 1.

Figure 1: Three-voter social network



The set of voters is equal to $N = \{1, 2, 3\}$, and both voters 1 and 2 influence voter 3. If voters 1 and 2 are unanimous with respect to their inclinations (that is, they have the same inclination), voter 3 will follow the influence of voters 1 and 2. Otherwise, voter 3 will follow his own inclination. Moreover, suppose that the group decision is a simple majority. We can write then:

$$b = (b_1, b_2, b_3) = \begin{cases} (i_1, i_2, i_1) \; if \; i_1 = i_2 \\ (i_1, i_2, i_3) \; if \; i_1 \neq i_2 \end{cases},$$

and

$$\forall i \in I \; [gd(Bi) = +1 \;\Leftrightarrow\; |\{k \in N \; |b_k = 1\}| \geq 2],$$

where $Bi = b = (b_1, b_2, b_3)$ denotes, of course, a decision vector.

Table 1 presents the group decision for our example.

Table 1: Group decision for Figure 1

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
$i^N = (1,1,1)$	$(1,1,1)$	$+1$	$i^{\emptyset} = (0,0,0)$	$(0,0,0)$	-1
$i^{(1)} = (1,1,0)$	$(1,1,1)$	$+1$	$i^{(6)} = (0,0,1)$	$(0,0,0)$	-1
$i^{(2)} = (1,0,1)$	$(1,0,1)$	$+1$	$i^{(5)} = (0,1,0)$	$(0,1,0)$	-1
$i^{(3)} = (0,1,1)$	$(0,1,1)$	$+1$	$i^{(4)} = (1,0,0)$	$(1,0,0)$	-1

Suppose now that we have a certain probability distribution p over all eight inclination vectors. Using the notations from Section 4, we get the following:

$$\Gamma_1(gd(B), p) = p(i^N) + p(i^{(1)}) + p(i^{(2)}) - p(i^{(3)}) - p(i^{(4)}) + p(i^{(5)}) + p(i^{(6)}) + p(i^{\emptyset})$$

$$\Gamma_2(gd(B), p) = p(i^N) + p(i^{(1)}) - p(i^{(2)}) + p(i^{(3)}) + p(i^{(4)}) - p(i^{(5)}) + p(i^{(6)}) + p(i^{\emptyset})$$

$$\Gamma_3(gd(B), p) = p(i^N) - p(i^{(1)}) + p(i^{(2)}) + p(i^{(3)}) + p(i^{(4)}) + p(i^{(5)}) - p(i^{(6)}) + p(i^{\emptyset}).$$

Next, we like to check when modification Γ_k coincides with Φ_k, that is, when Γ_k measures *decisiveness* of voter $k \in N$. By virtue of Proposition 1, we get

$$\begin{cases} \Gamma_1(gd(B), p) = \Phi_1(W, p) \\ \Gamma_2(gd(B), p) = \Phi_2(W, p) \\ \Gamma_3(gd(B), p) = \Phi_3(W, p) \end{cases} \Leftrightarrow \begin{cases} p(i^N) - p(i^{(3)}) = p(i^{(4)}) - p(i^{\emptyset}) \\ p(i^N) - p(i^{(2)}) = p(i^{(5)}) - p(i^{\emptyset}) \\ p(i^N) - p(i^{(1)}) = p(i^{(6)}) - p(i^{\emptyset}) \end{cases} \Leftrightarrow$$

$$p(i^N) + p(i^\emptyset) = p(i^{(3)}) + p(i^{(4)}) = p(i^{(2)}) + p(i^{(5)}) = p(i^{(1)}) + p(i^{(6)}) = \frac{1}{4}. \qquad (48)$$

For each probability distribution p which satisfies (48) we have

$$\mathit{\Gamma}_k(gd(B), p) = \mathit{\Phi}_k(W, p) = \frac{1}{2} \; for \; each \; k \in \{1, 2, 3\}.$$

One of the probability distributions satisfying (48) is, of course, p^* defined by equation (10).

We also have

$$\mathit{\Psi}_1(gd(B), p) = p(i^{(1)}) + p(i^{(5)}) + p(i^{(2)}) + p(i^{(6)})$$

$$\mathit{\Psi}_2(gd(B), p) = p(i^{(1)}) + p(i^{(4)}) + p(i^{(3)}) + p(i^{(6)})$$

$$\mathit{\Psi}_3(gd(B), p) = p(i^{(2)}) + p(i^{(4)}) + p(i^{(3)}) + p(i^{(5)})$$

For each probability distribution p which satisfies (48) we have, of course,

$$\mathit{\Psi}_k(gd(B), p) = \frac{1}{2} \; for \; each \; k \in \{1, 2, 3\}.$$

Next, we calculate when $\mathit{\Psi}_k(gd(B), p)$ coincides with the Shapley-Shubik index $Sh_k(W)$. By virtue of (31) and (32), we get $\mathit{\Psi}_k(gd(B), p) = Sh_k(W)$ for each $k = 1, 2, 3$ iff

$$\frac{1}{6} \le p(i^\emptyset) \le \frac{1}{3}$$

$$p(i) = \begin{cases} \frac{1}{3} - p(i^\emptyset) \; for \; i \in \{i^{(4)}, i^{(5)}, i^{(6)}\} \\ p(i^\emptyset) - \frac{1}{6} \; for \; i \in \{i^{(1)}, i^{(2)}, i^{(3)}\} \\ \frac{1}{2} - p(i^\emptyset) \; for \qquad i = i^N \end{cases}.$$

We have then

$$\mathit{\Psi}_k(gd(B), p) = Sh_k(W) = \frac{1}{3} \; for \; each \; k = 1, 2, 3.$$

Moreover,

$$\mathit{\Psi}_1^+(gd(B), p) = \frac{p(i^{(1)}) + p(i^{(2)})}{p(i^N) + p(i^{(1)}) + p(i^{(2)}) + p(i^{(4)})}$$

$$\mathit{\Psi}_2^+(gd(B), p) = \frac{p(i^{(1)}) + p(i^{(3)})}{p(i^N) + p(i^{(1)}) + p(i^{(3)}) + p(i^{(5)}))}$$

$$\mathit{\Psi}_3^+(gd(B), p) = \frac{p(i^{(2)}) + p(i^{(3)})}{p(i^N) + p(i^{(2)}) + p(i^{(3)}) + p(i^{(6)})}$$

By virtue of Proposition 4, $\mathit{\Psi}_k^+(gd(B), p) = Sh_k(W)$ for $k = 1, 2, 3$ iff

$$0 \le p(i^\emptyset) < 1$$

$$p(i) = \begin{cases} \frac{2}{11} \cdot (1 - p(i^\emptyset)) \; for \; i \in \{i^{(4)}, i^{(5)}, i^{(6)}, i^N\} \\ \frac{1}{11} \cdot (1 - p(i^\emptyset)) \; for \quad i \in \{i^{(1)}, i^{(2)}, i^{(3)}\} \end{cases},$$

and then

$$\Psi_k^+(gd(B),p) = Sh_k(W) = \frac{1}{3}\ for\ each\ k = 1,2,3.$$

Finally, we have

$$\Psi_1^-(gd(B),p) = \frac{p(i^{(5)}) + p(i^{(6)})}{p(i^{(3)}) + p(i^{(5)}) + p(i^{(6)}) + p(i^{\emptyset})}$$

$$\Psi_2^-(gd(B),p) = \frac{p(i^{(4)}) + p(i^{(6)})}{p(i^{(2)}) + p(i^{(4)}) + p(i^{(6)}) + p(i^{\emptyset})}$$

$$\Psi_3^-(gd(B),p) = \frac{p(i^{(4)}) + p(i^{(5)})}{p(i^{(1)}) + p(i^{(4)}) + p(i^{(5)}) + p(i^{\emptyset})}.$$

By virtue of Proposition 5, $\Psi_k^-(gd(B),p) = Sh_k(W)$ for $k = 1,2,3$ iff

$$0 \leq p(i^N) < 1$$

$$p(i) = \begin{cases} \frac{1}{11} \cdot (1 - p(i^N))\ for\ \ i \in \{i^{(4)}, i^{(5)}, i^{(6)}\} \\ \frac{2}{11} \cdot (1 - p(i^N))\ for\ i \in \{i^{\emptyset}, i^{(1)}, i^{(2)}, i^{(3)}\} \end{cases}.$$

Then we also have

$$\Psi_k^-(gd(B),p) = Sh_k(W) = \frac{1}{3}\ for\ each\ k = 1,2,3.$$

Finally, we want to calculate when $\Psi_k(gd(B),p)$ coincides with the non-normalized Holler-Packel index. In our example

$$I^{mw} = \{i^{(1)}, i^{(2)}, i^{(3)}\},$$

and hence, by virtue of Proposition 6, we get $\Psi_k(gd(B),p^{mw}) = HP_k(W)$ for $k = 1,2,3$, where

$$p^{mw}(i) = \begin{cases} \frac{1}{3}\ if\ \ \ i \in \{i^{(1)}, i^{(2)}, i^{(3)}\} \\ 0\ if\ i \in I \setminus \{i^{(1)}, i^{(2)}, i^{(3)}\} \end{cases}.$$

Then, we get

$$\Psi_k(gd(B),p^{mw}) = HP_k(W) = \frac{2}{3}\ for\ each\ k = 1,2,3.$$

8 Conclusion

In this paper, we consider measures of power for a social network in which players have either to accept or to reject a proposal. Each voter has an inclination either to say 'yes' or 'no', but he may also be influenced by other voter(s), and consequently his final decision may be different from his (original) inclination. To the best of our knowledge, the Hoede-Bakker index was analyzed so far only under the assumption of equally probable inclination vectors. The main aim of this paper is to generalize the (generalized) Hoede-Bakker index by considering arbitrary probability distributions over all inclination vectors. First, we define several modifications of the generalized decisional power index for an arbitrary probability distribution. Next, by choosing 'proper' probability distributions, we construct measures that coincide with the Shapley-Shubik index and with the Holler-Packel index. Finally, in order to illustrate the notions introduced, we consider a simple example.

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